# Mining (maximal) span-cores from temporal networks

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# Agenda

1. Temporal networks
2. Background and related work
3. Span-core decomposition
4. Maximal span-cores
5. Experiments
6. Applications
7. Conclusions

# Temporal networks

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- a temporal network is a representation of
  - **entities** (vertices)
  - their **relations** (links)
  - how these relations are **established/broken along time**

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- a temporal network is a representation of
  - **entities** (vertices)
  - their **relations** (links)
  - how these relations are **established/broken along time**
- extracting **dense structures** together with their **temporal span** is a key mining primitive
  - quantify the transmission opportunities of respiratory infections
  - identify events and buzzing stories
  - understand the dynamics of collaboration in successful professional teams

# Temporal graphs

## Definition

*A* **temporal graph** *is a triplet* $G = (V, T, \tau)$, *where*

- $V$ *is a set of vertices,*
- $T = [t_0, t_1, \ldots, t_{max}] \subseteq \mathbb{N}$ *is a discrete time domain,*
- $\tau : V \times V \times T \rightarrow \{0, 1\}$ *is a function defining for each* $u, v \in V$ *and each* $t \in T$ *whether edge* $(u, v)$ *exists in* $t$.

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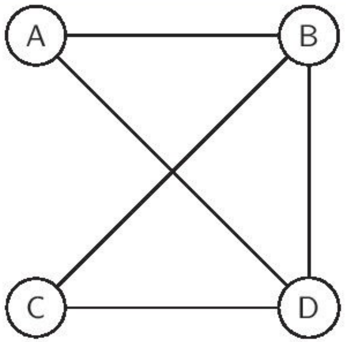
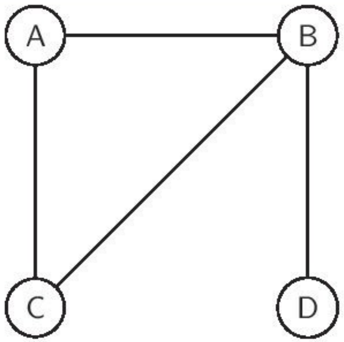



$t_0$




$t_1$

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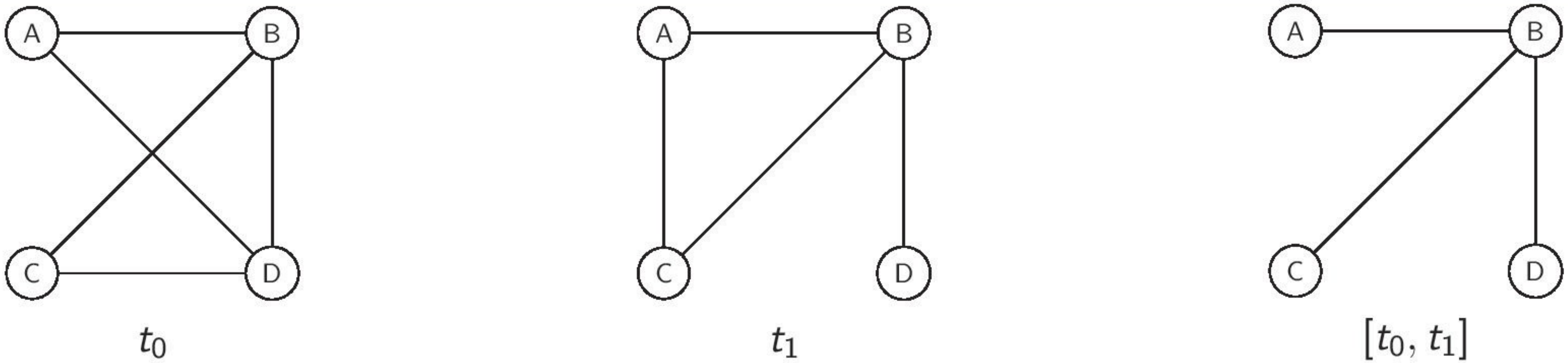
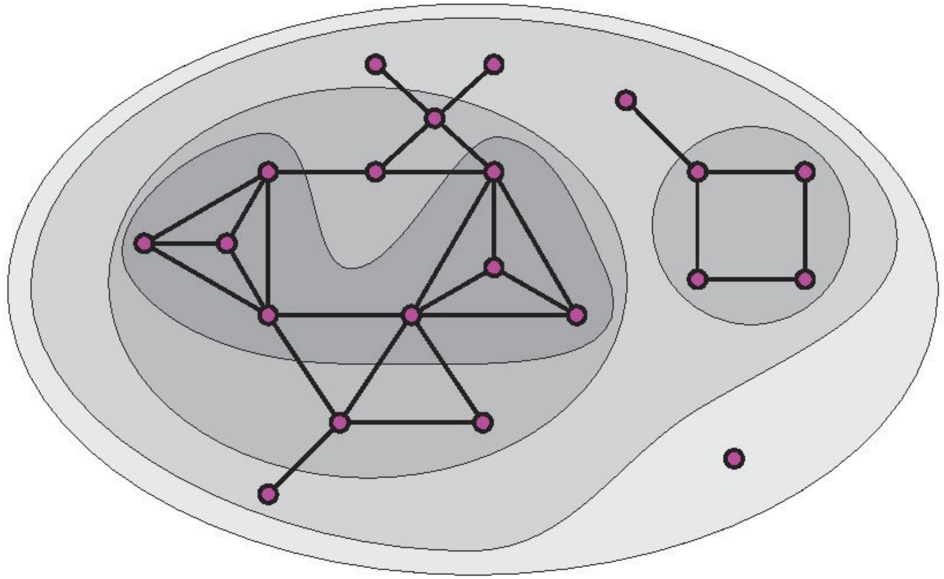


# Background and related work

# Core decomposition

## Definition

*The* **k-core** *(or core of order $k$) of a simple graph $G = (V, E)$ is a maximal set of vertices $C_k \subseteq V$ such that $\forall u \in C_k : deg(C_k, u) \geq k$.*
*The set of all k-cores $V = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_{k^*}$ is the* **core decomposition** *of $G$.*



# Core decomposition in multilayer networks [Galimberti *et al.* 2017]

## Definition

*Given a multilayer graph* $G = (V, E, L)$ *and an* $|L|$*-dimensional integer vector* $\vec{k} = [k_\ell]_{\ell \in L}$*, the* **multilayer $\vec{\mathbf{k}}$-core** *of* $G$ *is a maximal set of vertices* $C_{\vec{k}} \subseteq V$ *such that* $\forall u \in C_{\vec{k}}, \forall \ell \in L : deg(C_k, \ell, u) \geq k_\ell$*.*
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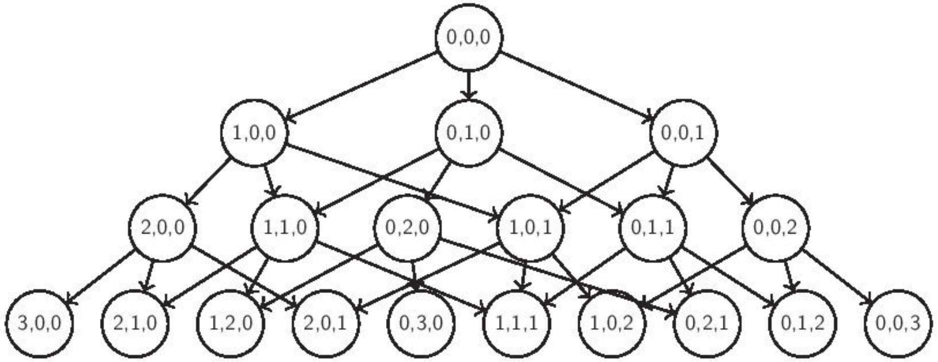


- the number of multilayer cores is **exponential** in the number of layer
- layers are not ordered, **there is not sequentiality**

# Core decomposition in complex networks

- *Core Decomposition of Uncertain Graphs* [Bonchi *et al.* 2014]
  - defined the $(k, \eta)$-core as the largest subgraph in which **the probability that every vertex has degree no less than k is greater or equal to** $\eta$
- *Core decomposition in large temporal graphs* [Wu *et al.* 2015]
  - defined the $(k, h)$-core as the largest subgraph in which every vertex has at least **k neighbors** and at least **h temporal connections** with each of them
  - **the sequentiality of connections is not taken into account**: non-contiguous timestamps can support the same core
- *When engagement meets similarity: efficient* $(k, r)$*-core computation on social networks* [Zhang *et al.* 2017]
  - studied the problem of **enumerating all maximal cores** of a (non-temporal) variant of core decomposition
  - **the problem is NP-hard**

# Span-core decomposition

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## Definition

*The* $(\mathbf{k}, \mathbf{\Delta})$-**core** *of a temporal graph* $G = (V, T, \tau)$ *is a maximal and non-empty set of vertices* $\emptyset \neq C_{k,\Delta} \subseteq V$*, such that* $\forall u \in C_{k,\Delta} : deg_{\Delta}(C_{k,\Delta}, u) \geq k$*, where* $\Delta \sqsubseteq T$ *is a temporal interval and* $k \in \mathbb{N}^{+}$*.*

- $deg_{\Delta}(C_{k,\Delta}, u)$ represents the **degree of a vertex** $u$ **in the subgraph induced by** $C_{k,\Delta}$ **within the temporal interval** $\Delta$

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## Problem

*Given a temporal graph* $G$*, find the set of all* $(k, \Delta)$*-cores of* $G$*.*

- the number of span-cores is $\mathcal{O}(|T|^2)$

# A naïve approach

## Algorithm

- generate all temporal intervals $\Delta \sqsubseteq T$
- for each $\Delta \sqsubseteq T$, compute the subgraph $G_\Delta = (V, E_\Delta)$
- run a core-decomposition subroutine on each $G_\Delta$

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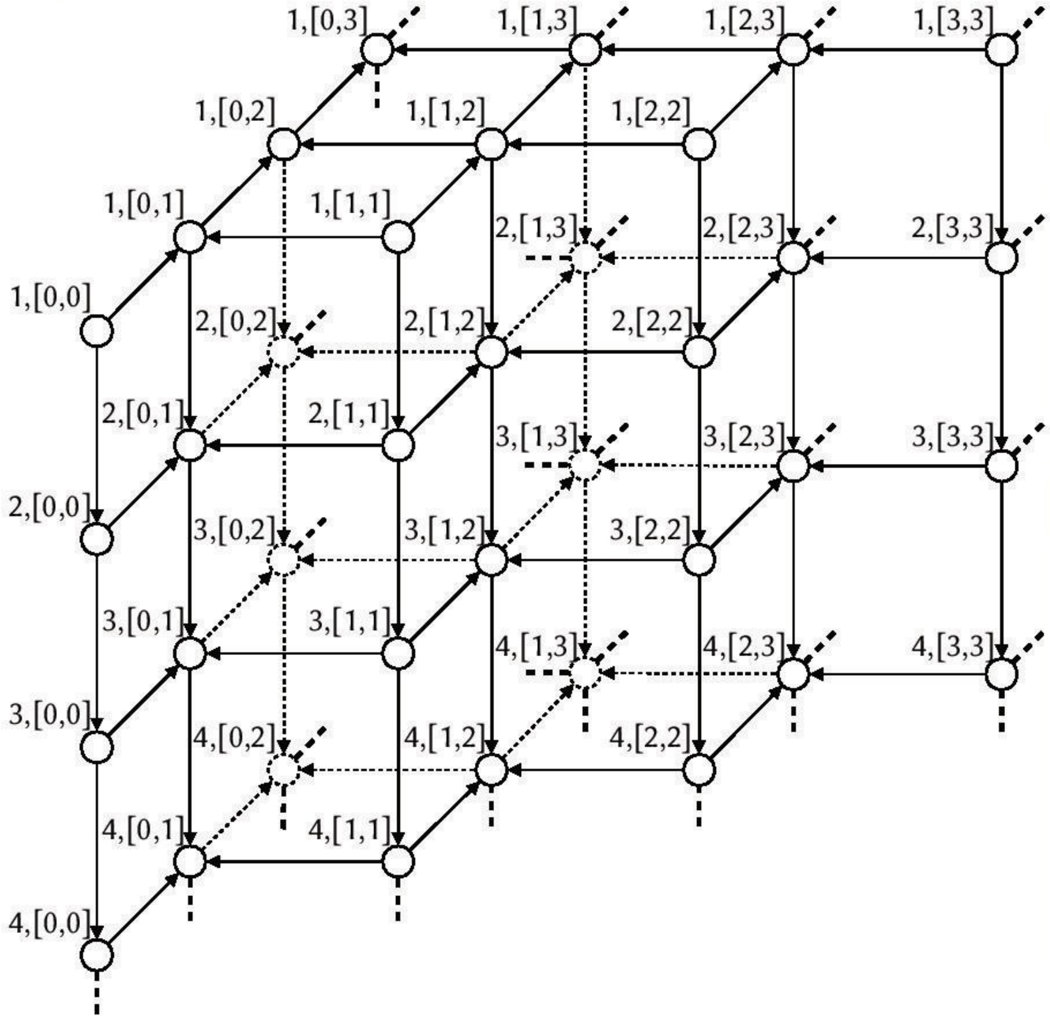
# Span-core search space




## Proposition

*For any two span-cores $C_{k,\Delta}$, $C_{k',\Delta'}$ of a temporal graph $G$ it holds that*

$$k' \leq k \wedge \Delta' \sqsubseteq \Delta \;\Rightarrow\; C_{k,\Delta} \subseteq C_{k',\Delta'}.$$

## Corollary

*Given a temporal graph $G = (V, T, \tau)$, and a temporal interval $\Delta = [t_s, t_e] \sqsubseteq T$, let $\Delta_+ = [\min\{t_s + 1, t_e\}, t_e]$ and $\Delta_- = [t_s, \max\{t_e - 1, t_s\}]$. It holds that*

$$C_{k,\Delta} \subseteq (C_{k,\Delta_+} \cap C_{k,\Delta_-}) = \bigcap_{\Delta' \sqsubseteq \Delta} C_{k,\Delta'}.$$

# A more efficient algorithm

## Algorithm

- generate temporal intervals $\Delta \sqsubseteq T$ of **increasing** size
- for each $\Delta \sqsubseteq T$ such that $|\Delta| > 1$, run a core-decomposition subroutine from $(C_{1,\Delta_+} \cap C_{1,\Delta_-})$
- if $C_{1,\Delta_+}$ or $C_{1,\Delta_-}$ does not exist, skip the core decomposition for $\Delta$

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**Definition**

*A span-core $C_{k,\Delta}$ of a temporal graph $G$ is said* **maximal** *if there does not exist any other span-core $C_{k',\Delta'}$ of $G$ such that $k \leq k'$ and $\Delta \sqsubseteq \Delta'$.*

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**Problem**

*Given a temporal graph $G$, find the set of all maximal $(k, \Delta)$-cores of $G$.*

- the number of maximal span-cores is $\mathcal{O}(|T|^2)$
- experimentally, maximal span-cores are **at least one order of magnitude less** than the overall span-cores

# A filtering approach

## Algorithm

- equip the algorithm for span-core decomposition with a data structure $\mathcal{M}$ that
  - stores the span-core of the highest order for every temporal interval $\Delta \sqsubseteq T$
  - at the storage of a span-core $C_{k,\Delta}$, removes the span-cores dominated by $C_{k,\Delta}$
- return the span-cores retained by $\mathcal{M}$

# Properties of maximal span-cores

## Lemma

*Given a temporal graph* $G = (V, T, \tau)$, *let* $\mathbf{C}_M$ *be the set of all maximal span-cores of* $G$, *and* $\mathbf{C}_{\mathbf{inner}} = \{C_{k^*}[G_\Delta] \mid \Delta \sqsubseteq T\}$ *be the set of innermost cores of all graphs* $G_\Delta$. *It holds that* $\mathbf{C}_M \subseteq \mathbf{C}_{\mathbf{inner}}$.

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**Lemma**

*Given a temporal graph $G = (V, T, \tau)$, and three temporal intervals $\Delta = [t_s, t_e] \sqsubseteq T$, $\Delta' = [t_s - 1, t_e] \sqsubseteq T$, and $\Delta'' = [t_s, t_e + 1] \sqsubseteq T$. The innermost core $C_{k^*}[G_\Delta]$ is a maximal span-core of $G$ if and only if $k^* > \max\{k', k''\}$ where $k'$ and $k''$ are the orders of the innermost cores of $G_{\Delta'}$ and $G_{\Delta''}$, respectively.*

**Lemma**

*Given $G$, $\Delta$, $\Delta'$, $\Delta''$, $k'$, and $k''$ as in previous Lemma, let $\widetilde{V} = \{u \in V \mid deg_\Delta(V, u) > \max\{k', k''\}\}$, and let $C_{k^*}[G_\Delta[\widetilde{V}]]$ be the innermost core of $G_\Delta[\widetilde{V}]$. If $k^* > \max\{k', k''\}$, then $C_{k^*}[G_\Delta[\widetilde{V}]]$ is a maximal span-core; otherwise, no maximal span-core exists for $\Delta$.*

# Efficient maximal-span-core finding

## Algorithm

- consider intervals $\Delta = [t_s, t_e] \sqsubseteq T$, for increasing values of $t_s$ and decreasing values of $t_e$
  - e.g., with $t_{max} = 10$, $\{[0,10],[0,9],\ldots,[0,0],[1,10],[1,9],\ldots,[1,1],[2,10],[2,9],\ldots\}$
  - this guarantees that once we consider $\Delta$, no $\Delta' \sqsupseteq \Delta$ will be considered at later stage
- compute the **lower bound lb on the order** of a span-core in $\Delta$ to be recognized as maximal
- build the sets of vertices $V_{lb}$ that have degree in $\Delta$ larger than $lb$
- extract the **innermost** core of the subgraph $(V_{lb}, E_\Delta[V_{lb}])$
- identify such a core as maximal if its order is actually larger than $lb$

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- identify such a core as maximal if its order is actually larger than $lb$

- the time complexity is still $\mathcal{O}(|T|^2 \times |E|)$

# Experiments

# Datasets

| dataset | $\lvert V \rvert$ | $\lvert E \rvert$ | $\lvert T \rvert$ | window size (days) | domain |
|---|---|---|---|---|---|
| ProsperLoans | 89k | 3M | 307 | 7 | economic |
| Last.fm | 992 | 4M | 77 | 21 | co-listening |
| WikiTalk | 2M | 10M | 192 | 28 | communication |
| DBLP | 1M | 11M | 80 | 366 | co-authorship |
| StackOverflow | 2M | 16M | 51 | 56 | question-and-answer |
| Wikipedia | 343k | 18M | 101 | 56 | co-editing |
| Amazon | 2M | 22M | 115 | 28 | co-rating |
| Epinions | 120k | 33M | 25 | 21 | co-rating |

# Evaluation

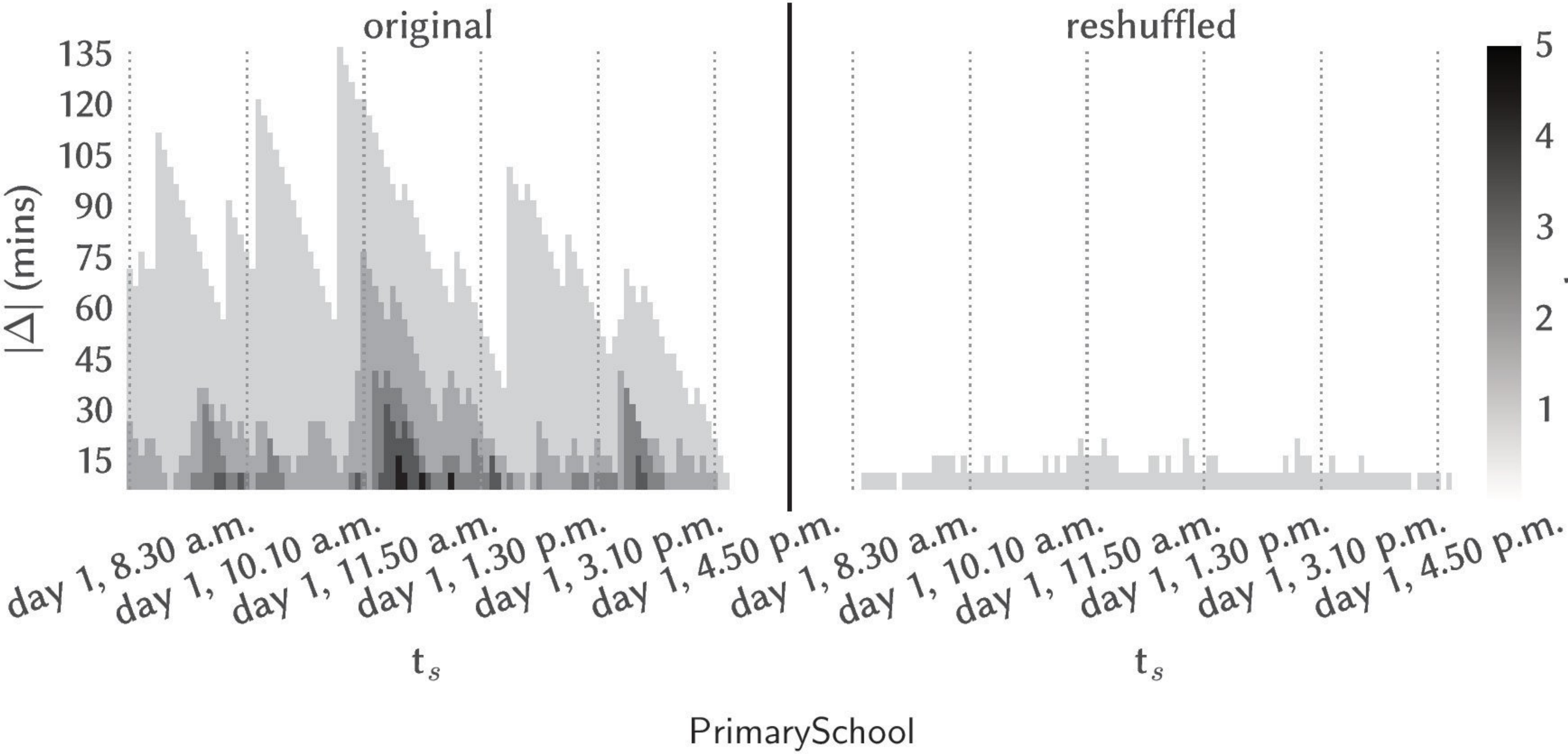
| dataset | method | # output span-cores | time (s) | memory (GB) | # processed vertices |
|---|---|---|---|---|---|
| WikiTalk | Naïve-span-cores | 19 693 | 322 302 | 36 | 25B |
| | Span-cores | | 1 084 | 36 | 555M |
| | Naïve-maximal-span-cores | 632 | 1 194 | 36 | 555M |
| | Maximal-span-cores | | 126 | 35 | 2M |
| Wikipedia | Naïve-span-cores | 125 191 | 17 155 | 4 | 1B |
| | Span-cores | | 522 | 4 | 35M |
| | Naïve-maximal-span-cores | 2 147 | 537 | 4 | 35M |
| | Maximal-span-cores | | 201 | 4 | 320k |
| Amazon | Naïve-span-cores | 29 318 | 10 415 | 18 | 2B |
| | Span-cores | | 409 | 18 | 247M |
| | Naïve-maximal-span-cores | 303 | 580 | 18 | 247M |
| | Maximal-span-cores | | 123 | 18 | 688k |

# Applications

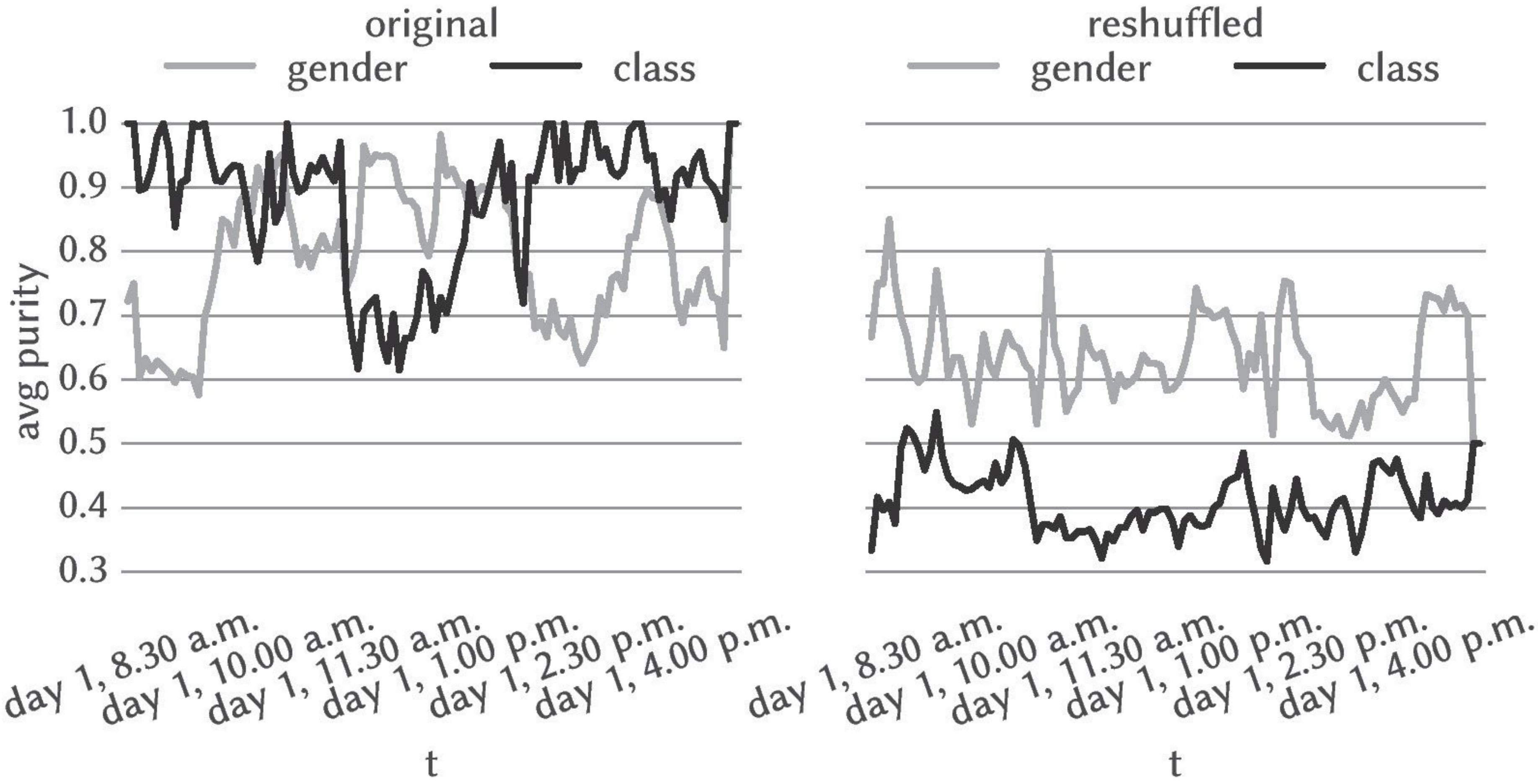
# Datasets

- **face-to-face interaction networks** gathered by a proximity-sensing infrastructure in schools
  - PrimarySchool (242 individuals, 2 days)
  - HighSchool (327 individuals, 5 days)
  - HongKong (774 individuals, 11 days)
- window size of 5 minutes
- discarded span-cores of $|\Delta| = 1$

# Social activities of groups of students

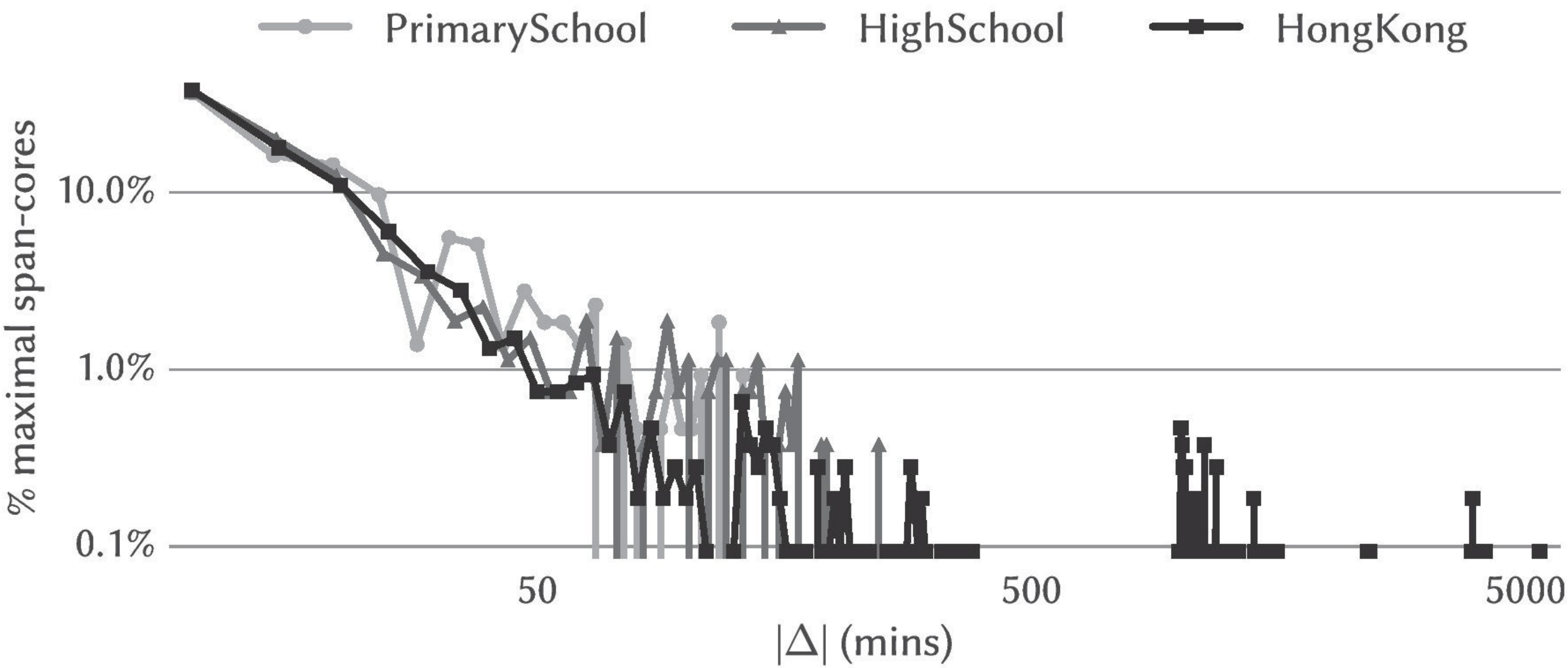


# Mixing of gender and class

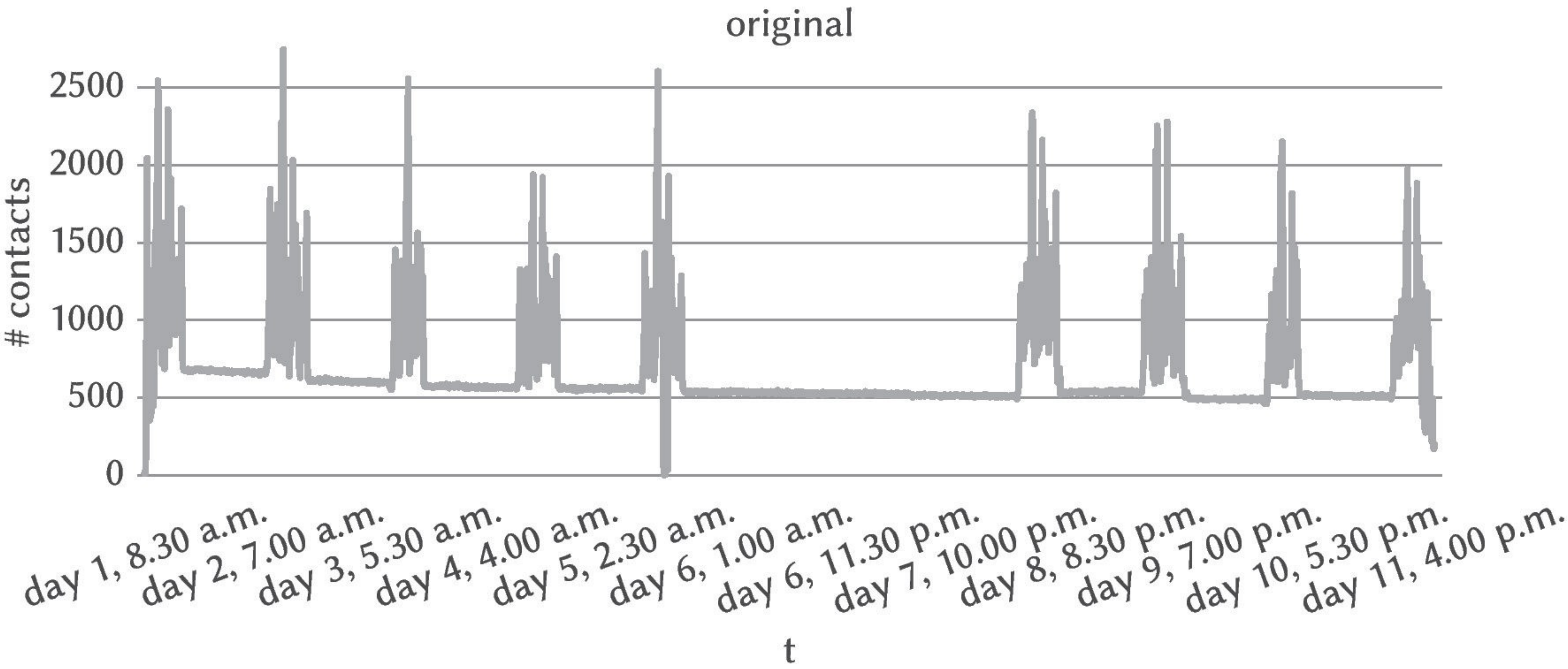



PrimarySchool

# Length of social interactions in groups



# Anomaly detection




HongKong

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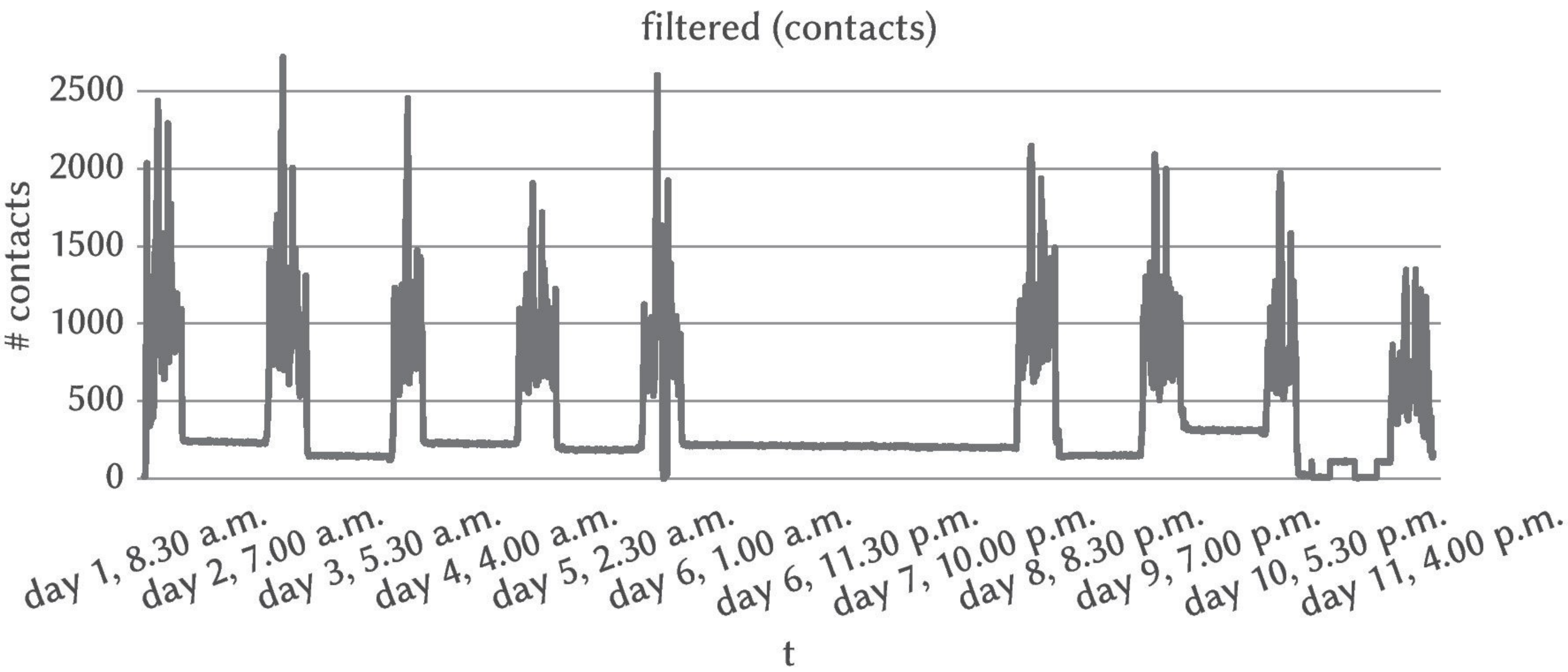


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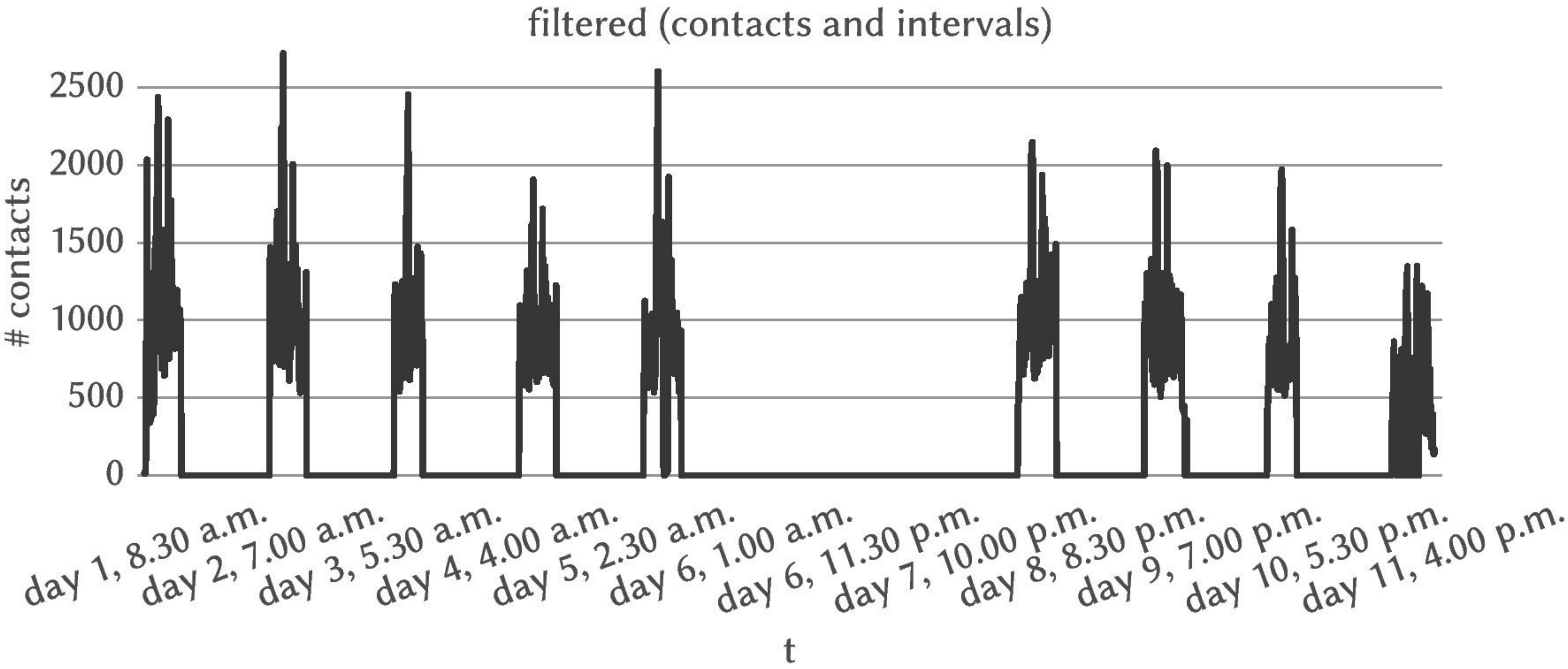


HongKong

# Conclusions

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- introduced a notion of dense pattern in temporal networks that
  - takes into account the **sequentiality** of connections
  - is assigned with a clear **temporal collocation**
- developed efficient algorithms for computing all the span-cores, and only the maximal ones
- future work:
  - spreading processes analysis
  - temporal community search and temporal densest subgraph
  - network finger-printing

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